EMERGENCE OF TAYLOR VORTICES BETWEEN ROTATING ECCENTRIC CYLINDERS

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In a linear setting we examine the stability of the flow of a viscous incompressible liquid between eccentric cylinders, the inner cylinder rotating and the outer cylinder fixed. We consider the case of a narrow gap between the cylinders, which is characteristic for problems in lubrication theory. The main flow, perturbations, and the critical Reynolds number are found in the form of expansions in powers of the eccentricity ε to within $O(\varepsilon^3)$. The results obtained agree with the known experimental data for $0 \leq \varepsilon \leq 0.5$ and confirm the stabilizing influence of the eccentricity.

In the experimental papers [1-5] it is shown that during the loss of stability of Couette flow between rotating eccentric cylinders, a secondary flow of Taylor vortex type arises. Attempts to analyze this phenomenon by the method of "local stability," i.e., the investigation of the stability of velocity profiles at various sections of the gap between the cylinders, has led to contradictory results [6-8].

From the point of view of the calculation of bearing slippage, the case in which there is a small gap ψ between the cylinders is of particular interest [$\psi = (R_2 - R_1)/R_2$, R_1 and R_2 are the cylinder radii]. Writing the equations of motion of the viscous liquid in bipolar coordinates and discarding them in terms of higher order in ψ, we obtain the following system of equations, a detailed derivation of which is given in [7]:

$$\frac{\partial^2 V}{\partial y^2} - h^2 \frac{\partial P}{\partial \varphi} = \mathrm{Re}^* h \left(hV \frac{\partial V}{\partial \varphi} - U \frac{\partial V}{\partial y} \right)$$
$$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial U}{\partial y} - \frac{\partial}{\partial \varphi}(hV) = 0 \quad (h = 1 + \varepsilon \cos \varphi) \tag{1}$$

In Eqs. (1) the velocity components V and U are referred to ωR_1 (ω is the angular rotation rate of the inner cylinder), the pressure P is referred to $\rho \omega^2 R_1^2 \mathrm{Re}^{*-1}$ [$\mathrm{Re}^* = \mathrm{Re}\,\psi$, $\mathrm{Re} = \omega R_1 (R_2 - R_1)/\nu$ is the Reynolds number, ρ is the density, and ν is the viscosity of the liquid].

The boundary conditions for the system (1) are

$$V(0, \varphi) = U(0, \varphi) = U(1, \varphi) = 0, \quad V(1, \varphi) = 1 \tag{2}$$

The coordinate y is reckoned from the outer cylinder and is equal to 1 on the inner cylinder. For the functions U, V, and P periodicity conditions in φ must also be satisfied.

The study of the stability of the flow (V, U) by the method of small perturbations reduces to a characteristic value problem for the number Re (see [7]):

$$\left(\frac{\partial^2}{\partial y^2} - k^2 h^2 \right)^2 u = 2k^2 \mathrm{Re}^* h^4 \left\{ Vv + \frac{1}{2h} \frac{\partial}{\partial y}(Uu) \right.$$
$$\left. - \frac{V}{2h} \frac{\partial}{\partial \varphi}(hu) + \frac{1}{2k^2 h^2} \frac{\partial}{\partial y} \left[hV \frac{\partial}{\partial \varphi} - U \frac{\partial}{\partial y} \right] \left[\frac{1}{h} \frac{\partial u}{\partial y} \right] \right\} \tag{3}$$
$$\left(\frac{\partial^2}{\partial y^2} - k^2 h^2 \right) v = -\mathrm{Re}\, h \frac{\partial V}{\partial y} u, \quad u = \frac{\partial u}{\partial y} = v = 0 \quad (y = 0, 1)$$

with periodicity conditions in φ for u, v, and p.

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Here u and v are perturbation velocity components and k is the wave number. We consider the class of perturbations corresponding to observed Taylor vortices, i.e., perturbations which for $\varepsilon = 0$ become rotationally symmetric. We postulate, as in the case of concentric cylinders, that the "principle of stability variation" applies.

In studying stability we consider the problem (1), (2) as an auxiliary problem even through it is of interest in its own right (see [9-12]). We seek a solution of this problem in the form of series in powers of ε:

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots, \quad V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \ldots$$
$$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots \tag{4}$$

The convergence of these series for sufficiently small ε follows from the results given in [13].

Substituting Eqs. (4) into Eqs. (1) and (2), we obtain

$$V_0 = y, \ U_0 \equiv 0, \qquad P_0 \equiv \text{const}$$
$$\frac{\partial^2 V_1}{\partial y^2} - \frac{\partial P_1}{\partial \varphi} = \text{Re}^*\left(y\frac{\partial V_1}{\partial \varphi} - U_1\right), \quad \frac{\partial P_1}{\partial y} = 0 \tag{5}$$

$$\frac{\partial U_1}{\partial y} - \frac{\partial V_1}{\partial \varphi} + y \sin\varphi = 0, \quad U_1 = V_1 = 0 \quad (y = 0, 1) \tag{6}$$

$$\frac{\partial^2 V_2}{\partial y^2} - \frac{\partial P_2}{\partial \varphi} = \text{Re}^*\left(y\frac{\partial V_2}{\partial \varphi} - U_2\right) + 2\cos\varphi\frac{\partial P_1}{\partial \varphi} + \text{Re}^*\left[V_1\frac{\partial V_1}{\partial \varphi} + U_1\frac{\partial V_1}{\partial y} + \cos\varphi\left(2y\frac{\partial V_1}{\partial \varphi} - U_1\right)\right] \tag{7}$$

$$\frac{\partial P_2}{\partial y} = 0, \quad \frac{\partial U_2}{\partial y} - \frac{\partial V_2}{\partial \varphi} - \frac{\partial}{\partial \varphi}(V_1\cos\varphi) = 0, \quad U_2 = V_2 = 0 \quad (y = 0, 1)$$

Writing the solutions of the problems (6) and (7) in the form of Fourier series in φ, we confirm that they are of the form

$$\begin{aligned} V_1 &= V_{11}(y)\sin\varphi + V_{12}(y)\cos\varphi \\ U_1 &= U_{11}(y)\sin\varphi + U_{12}(y)\cos\varphi \\ P_1 &= P_{11}\sin\varphi + P_{12}\cos\varphi \end{aligned} \tag{8}$$

$$\begin{aligned} V_2 &= V_{20}(y) + V_{21}(y)\sin 2\varphi + V_{22}(y)\cos 2\varphi \\ U_2 &= U_{21}(y)\sin 2\varphi + U_{22}(y)\cos 2\varphi \\ P_2 &= P_{20} + P_{21}\sin 2\varphi + P_{22}\cos 2\varphi \end{aligned} \tag{9}$$

Problem (6) reduces to a system of ordinary differential equations

$$\begin{gathered} \bar{U}_{11}^{\text{IV}} + \text{Re}^*\, y\, U_{12}'' = 0, \ \bar{U}_{11}(0) = \bar{U}_{11}'(0) = \bar{U}_{11}(1) = \bar{U}_{11}'(1) = 0 \\ U_{12}^{\text{IV}} - \text{Re}^*\, y\, \bar{U}_{11}'' = \text{Re}^*(4y - 6y^2), \ \ U_{12}(0) = U_{12}'(0) = U_{12}(1) \\ = U_{12}'(1) = 0 \\ V_{11} = U_{12}', \ V_{12} = -y - U_{11}' \end{gathered} \tag{10}$$

where

$$\bar{U}_{11} = U_{11} - y^2(1 - y) \tag{11}$$

Similarly, problem (7) reduces to the system of equations

$$\begin{gathered} U_{21}^{\text{IV}} + 2\text{Re}^* y U_{22}'' = F_1, \ U_{21}(0) = U_{21}'(0) = U_{21}(1) = U_{21}'(1) = 0 \\ U_{22}^{\text{IV}} - 2\text{Re}^* y\, U_{21}'' = F_2, \ U_{22}(0) = U_{22}'(0) = U_{22}(1) = U_{22}'(1) = 0 \\ V_{21} = {}^1/_2\,(U_{22}' - V_{11}), \qquad V_{22} = -{}^1/_2\,(U_{21}' + V_{12}) \\ V_{20} = {}^1/_2\,\text{Re}^*(U_{11}V_{11}' + U_{12}V_{12}' + U_{12} + 2y\,V_{11}) + V_{12}''(0), \ V_{20}(0) \\ = V_{20}(1) = 0 \end{gathered} \tag{12}$$

Here

$$\begin{gathered} F_1 = \text{Re}^*\,[U_{12}V_{12}'' + U_{12}'V_{12}' + U_{11}V_{11}'' + U_{11}'V_{11}' + U_{12}' - \\ - 2V_{11}V_{12}' - 2V_{11}'V_{12} - y\,V_{11}'] \\ F_2 = \text{Re}^*\,[2V_{11}V_{11}' - 2V_{12}V_{12}' - U_{11}V_{12}'' - U_{11}'V_{12}' - \\ - U_{12}\,V_{11}'' - U_{12}'V_{11}' - U_{11}' - y\,V_{12}' + y] \end{gathered} \tag{13}$$

In solving the problems (10) and (12) we applied the method of moments [14] (a particular version of the Petrov–Galerkin method). The solution is sought in the form

$$\bar{U}_{11,n} = y^2(1-y)^2 \sum_{k=0}^{n} A_k y^k, \quad U_{12,n} = y^2(1-y)^2 \sum_{k=0}^{n} B_k y^k$$
$$U_{21,n} = y^2(1-y)^2 \sum_{k=0}^{n} a_k y^k, \quad U_{22,n} = y^2(1-y)^2 \sum_{k=0}^{n} b_k y^k \tag{14}$$

The constants A_k, B_k, a_k, b_k are determined from the systems of equations

$$\int_0^1 (\bar{U}^{IV}_{11,n} + \mathrm{Re}^*\, y U''_{12,n})\, y^k dy = 0$$
$$\int_0^1 (U^{IV}_{12,n} - \mathrm{Re}^*\, y \bar{U}''_{11,n})\, y^k dy = 6\,\mathrm{Re}^* / (3+k) \quad (k = 0, 1, \ldots, n) \tag{15}$$

$$\int_0^1 (U^{IV}_{21,n} + 2\,\mathrm{Re}^*\, y U''_{22,n})\, y^k dy = \int_0^1 F_1 y^k\, dy$$
$$\int_0^1 (U^{IV}_{22,n} - 2\,\mathrm{Re}^*\, y U''_{21,n})\, y^k\, dy = \int_0^1 F_2 y^k\, dy \tag{16}$$

We solved the algebraic systems of equations (15) and (16) for various values of Re* on the M-20 electronic digital calculator for various n to achieve the required accuracy. As a rule, convergence, for all practical considerations, was observed for n = 9.

For the solution of the stability problem (3) the characteristic Re number and the characteristic functions u and v corresponding to it can be written in the form of power series in ε for sufficiently small eccentricities:

$$\mathrm{Re} = \mathrm{Re}_0 + \varepsilon\, \mathrm{Re}_1 + \varepsilon^2 \mathrm{Re}_2 + \ldots \tag{17}$$

$$u(y, \varphi) = u_0(y) + \varepsilon u_1(y, \varphi) + \varepsilon^2 u_2(y, \varphi) + \ldots$$
$$v(y, \varphi) = v_0(y) + \varepsilon v_1(y, \varphi) + \varepsilon^2 v_2(y, \varphi) + \ldots \tag{18}$$

Subsitution of the expansions (4), (17), and (18) into Eqs. (3) leads to the following recursion problems:

$$(d^2/dy^2 - k^2)^2 u_0 - 2k^2\,\mathrm{Re}^*\, y\, v_0 = 0, \quad u_0 = du_0/dy = 0 \quad (y = 0, 1)$$
$$(d^2/dy^2 - k^2)\, v_0 + \mathrm{Re}_0 u_0 = 0, \quad v_0 = 0 \quad (y = 0, 1) \tag{19}$$

$$\left[\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 - \mathrm{Re}^* \frac{\partial}{\partial \varphi}\left(y \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} - k^2 y\right)\right] u_1 - 2k^2\,\mathrm{Re}^*\, y v_1 =$$
$$= 4k^2 \cos\varphi \left(\frac{d^2}{dy^2} - k^2\right) u_0 + \mathrm{Re}^* \left(y \frac{d^2 u_0}{dy^2} \sin\varphi + \frac{du_0}{dy} \sin\varphi - \right.$$
$$- U_1 \frac{d^3 u_0}{dy^3} - \frac{\partial U_1}{\partial y} \frac{d^2 u_0}{dy^2} + k^2 u_0 \frac{\partial U_1}{\partial y} + k^2 \frac{du_0}{dy} U_1$$
$$\left. + k^2 u_0 y \sin\varphi + 2k^2 v_0 V_1 + 8k^2 v_0 y \cos\varphi \right) \equiv \Phi_{11} \tag{20}$$

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right) v_1 + \mathrm{Re}_0 u_1 = 2k^2 v_0 \cos\varphi - \mathrm{Re}_0 u_0 \left(\frac{\partial V_1}{\partial y} + \cos\varphi\right) -$$
$$- \mathrm{Re}_1 u_0 \equiv \Phi_{12}, \quad u_1 = \partial u_1/\partial y = v_1 = 0 \quad (y = 0, 1)$$
$$\left[\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 - \mathrm{Re}^* \frac{\partial}{\partial \varphi}\left(y \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} - k^2 y\right)\right] u_2 - 2k^2\,\mathrm{Re}^*\, y v_2 = \Phi_{21}$$
$$(\partial^2/\partial y^2 - k^2)\, v_2 + \mathrm{Re}_0 u_2 = \Phi_{22}, \quad u_2 = \partial u_2/\partial y = v_2 = 0 \quad (y = 0, 1) \tag{21}$$

The functions u_1, v_1, and u_2, v_2 must also satisfy periodicity conditions in φ. The expressions for Φ_{21} and Φ_{22} are not given here on account of their complexity.

Problem (19) corresponds to the case of rotating concentric cylinders; its solution is well known (see, for example, [15]).

For solvability of the problems (20) and (21) it is necessary and sufficient that the following conditions be satisfied:

$$\int_0^{2\pi}\int_0^1 (\Phi_{11} u_0^* + \Phi_{12} v_0^*)\, dy\, d\varphi = 0 \tag{22}$$

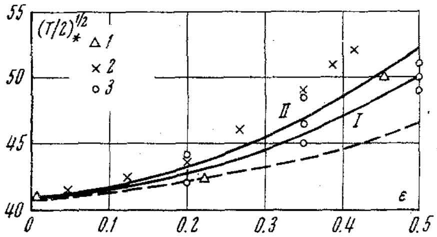

Fig. 1

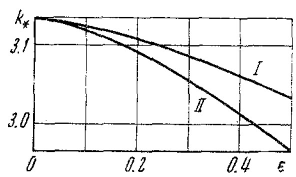

Fig. 2

$$\int_0^{2\pi}\int_0^1 (\Phi_{21}u_0{}^* + \Phi_{22}v_0{}^*)\,dy\,d\varphi = 0 \tag{23}$$

Here $u_0{}^*$ and $v_0{}^*$ are solutions of the adjoint problem

$$\begin{gathered}(d^2/dy^2 - k^2)^2 u_0{}^* + \mathrm{Re}_0 v_0{}^* = 0,\ u_0{}^* = du_0{}^*/dy = 0 \quad (y = 0, 1)\\ (d^2/dy^2 - k^2)\, v_0{}^* - 2k^2\mathrm{Re}^*\, y\, u_0{}^* = 0, \quad v_0{}^* = 0 \quad (y = 0, 1)\end{gathered} \tag{24}$$

It follows from the condition (22), after the expressions (8) have been substituted into Φ_{11} and Φ_{12}, that

$$\mathrm{Re}_1 = 0$$

Let $T_0 = 2\mathrm{Re}_0{}^2\psi$ be the Taylor number for the rotating concentric cylinders problem. If we seek u_0 and v_0 in the form

$$u_{0n} = y^2(1-y)^2 \sum_{k=0}^{n} C_k y^k, \quad v_{0n} = (2k^2\,\mathrm{Re}^*)^{-1}\, y\,(1-y) \sum_{k=0}^{n} D_k y^k \tag{25}$$

then the application of the moment method equations to problem (19) allows us to determine the smallest characteristic value T_0 as a function of the wave number k, where $T_{0\,\min} = 3389.9$ for $k = 3.13$ with $n = 6$. We can also determine the corresponding characteristic functions from Eqs. (25) and, similarly, the characteristic functions of the adjoint problem (24).

Putting $\mathrm{Re}_1 = 0$ in Eq. (20) and knowing u_0 and v_0, we determine functions u_1 and v_1, their Fourier series expansions being of the form

$$u_1 = u_{11}(y)\sin\varphi + u_{12}(y)\cos\varphi, \quad v_1 = v_{11}(y)\sin\varphi + v_{12}(y)\cos\varphi \tag{26}$$

Substitution of Eqs. (26) into Eqs. (20) leads to the system of equations

$$\begin{gathered}\left(\frac{d^2}{dy^2} - k^2\right)^2 u_{11} + \mathrm{Re}^*\left(y\frac{d^2}{dy^2} + \frac{d}{dy} - k^2 y\right) u_{12} - 2k^2\,\mathrm{Re}^*\, y v_{11} =\\ = \mathrm{Re}^*\left(U_{11}\frac{d^3u_0}{dy^3} - \frac{dU_{11}}{dy}\frac{d^2u_0}{dy^2} + y\frac{d^2u_0}{dy^2} + \frac{du_0}{dy} + \right.\\ \left. + k^2u_0\frac{dU_{11}}{dy} + k^2\frac{du_0}{dy}U_{11} + k^2 y v_0 + \frac{1}{\mathrm{Re}^*}V_{11}v_0\right)\\ \left(\frac{d^2}{dy^2} - k^2\right)^2 u_{12} - \mathrm{Re}^*\left(y\frac{d^2}{dy^2} + \frac{d}{dy} - k^2 y\right) u_{11} - 2k^2\,\mathrm{Re}^*\, y v_{12} =\\ = 4k^2\left(\frac{d^2}{dy^2} - k^2\right)u_0 + \mathrm{Re}^*\left(-U_{12}\frac{d^3u_0}{dy^3} - \frac{dU_{12}}{dy}\frac{d^2u_0}{dy^2} + \right.\\ \left. + k^2u_0\frac{dU_{12}}{dy} + k^2\frac{du_0}{dy}U_{12} + \frac{1}{\mathrm{Re}^*}V_{12}v_0 + \frac{4}{\mathrm{Re}^*}\, y v_0\right)\\ \left(\frac{d^2}{dy^2} - k^2\right)v_{11} + \frac{T_0}{2\,\mathrm{Re}^*}u_{11} = -\frac{T_0}{2\,\mathrm{Re}^*}u_0\frac{dV_{11}}{dy}\\ \left(\frac{d^2}{dy^2} - k^2\right)v_{12} + \frac{T_0}{2\,\mathrm{Re}^*}u_{12} = \frac{1}{\mathrm{Re}^*}v_0 - \frac{T_0}{2\,\mathrm{Re}^*}u_0\left(\frac{dV_{12}}{dy} + 1\right)\\ u_{11}(0) = u_{11}(1) = du_{11}/dy\,(0) = du_{11}/dy\,(1) = 0\\ u_{12}(0) = u_{12}(1) = du_{12}/dy\,(0) = du_{12}/dy\,(1) = 0\\ v_{11}(0) = v_{11}(1) = v_{12}(0) = v_{12}(1) = 0\end{gathered} \tag{27}$$

The functions u_{11}, u_{12}, v_{11}, v_{12} were determined by the moment method described above for fixed values of k and Re^*. Next we substituted the solutions of problems (19) and (20) for these same values of k and

Re^* into the expressions for Φ_{21} and Φ_{22} in the solvability condition (23), and from these we calculated the value of Re_2 (k, Re^*).

The results obtained were reduced to the form

$$(T / 2)^{1/2} = \mathrm{Re}\,\psi^{1/2} = (T_0 / 2 + \varepsilon^2 \,\mathrm{Re}_2\mathrm{Re}^* + \ldots)^{1/2} \tag{28}$$

For fixed values of Re^* and ε from the intervals $0.1 \leq Re^* \leq 9$, $0 \leq \varepsilon \leq 0.5$, we determined the value of $(T/2)_*^{1/2} = \min_k (T/2)^{1/2}$ and the corresponding critical wave number k_*. These results, together with the known experimental data, are shown in Figs. 1 and 2 (for curves I and II the values of Re^* were 0.1 and 9, respectively; for the data labeled 1, 2, and 3 values of ψ were, respectively, 0.08, 0.045, and 0.01, taken from [1], [3], and [2], respectively). The curves calculated for $0.1 < Re^* < 9$ are not shown here; if drawn, they would appear, in both figures, between the curves labeled I and II.

Thus, it follows from the linear theory of stability that for relatively small eccentricities there is a tendency for k_* to decreases as ε increases, while the critical Taylor number increases monotonically with an increase in the eccentricity. Moreover, an increase in the modified Reynolds number Re^* (equivalent to an increase in the relative gap ψ) further reinforces each of these effects.

Similar results were obtained recently by another method, and under additional assumptions (in particular, the wave number k was assumed to have the fixed value k=3.127), by R. Di Prima and J. Stuart [16, 17]. Their results are shown for comparison in Fig. 1 (dashed curve).

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